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COMPUTING p-SUMMING NORMS WITH FEW VECTORS

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ABSTRACT

It is shown that the p-summing norm of any operator with n-dimensional domain can be well-aproximated using only "few" vectors in the definition of the p-summing norm. Except for constants independent of n and $\log n$ factors, "few" means n if $1 < p < 2$ and $n^{p/2}$ if $2 < p < \infty$.

I. Introduction

A useful result of Tomczak-Jaegermann [T-J, p. 143] states that the 2-summing norm of an operator u of rank n can be well-estimated by n vectors; precisely (in the notation of [T-J, p. 140], which we follow throughout), $\pi_2(u) \leq \sqrt{2}\pi_2^{(n)}(u)$. No such result holds for π_1; Figiel and Pelczynski [T-J, p. 184] showed that if k_n satisfies $\pi_1(u) \leq C\pi_1^{(k_n)}(u)$ for all operators of rank n; $n = 1, 2, \ldots$, then k_n grows exponentially in n. The Tomczak result reduces immediately to the

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case of operators whose domains are ℓ_2^n. Szarek [Sz] proved that there is a 1-summing analogue to this version of Tomczak's theorem; namely, that $\pi_1(u) \le C\pi_1^{(n\log n)}(u)$ whenever u is an operator whose domain has dimension n.

In this paper we consider the case of p-summing operators. In section III we extend Szarek's result to the range $1 < p < 2$ (except that the power of $\log n$ is "3" instead of "1"). For $2 < p < \infty$ we show that, up to powers of $\log n$, $n^{\frac{p}{2}}$ vectors suffice to well-estimate the p-summing norm of an operator from an n-dimensional space. The power of n is optimal, but we do not know whether a $\log n$ term is needed in either result. These results, as well as those in section IV, shed some light on problems 24.10, 24.11, and 24.6 in [T-J].

In Section IV we show that when $1 < p \ne 2 < \infty$, if k_n satisfies $\pi_p(u) \le C\pi_p^{(k_n)}(u)$ for all operators of rank n; $n = 1, 2, \dots$, then k_n grows faster than any power of n. .

Just as for Szarek, our main tools are sophisticated versions of embedding n-dimensional subspaces of L_p into ℓ_p^k with k not too large. While most of this background is at least implicit in [BLM] and [T], we need more precise versions of such results than are stated in the current literature. The necessary material is developed in Section II.

Here we treat only the case of p-summing operators. There is also an extensive literature on related problems for (p,q)-summing operators; see [T-J] for the older history and the recent papers [DJ1], [DJ2], [DJ3], [J]. In particular, Defant and Junge [DJ2] show how results for p-summing operators can be formally transformed into results for (p,q)-summing operators.

II. Preparations for the main result

Before stating the basic entropy lemma for the main result, we set some notation.

A **density** on a probability space (Ω,μ) is a strictly positive measurable function on Ω whose integral is one. Given a set A, a metric δ on A, and a positive number t, $E(A,\delta,t)$ is the minimal number of open balls of radius t in the metric δ needed to cover A.

We also use notation (see, for example, [T-J, p. 80]) commonly used in Banach space theory for measuring the expected value of the norm of Gaussian processes: If $u\colon H \to Z$ is a linear operator from a finite dimensional Hilbert space H into a normed space Z, $\ell(u)^2$ is defined to be $\mathbb{E}\|\sum_{i=1}^m g_i u(e_i)\|^2$, where $e_1,\dots,e_m$ is any orthonormal basis for H and $g_1,\dots,g_m$ are independent standard Gaussian variables. ℓ is an ideal norm in the sense that if H' is another finite dimensional Hilbert space, Z' is another normed space, $T\colon H' \to Z$ and $S\colon Z \to Z'$ are linear operators, then $\ell(SuT) \le \|S\|\ell(u)\|T\|$. Suppose now that ν is a probability measure on a finite set A and W is an n-dimensional subspace of the set of scalar valued functions on A. Let W_p denote W under the $L_p(\nu)$–norm and let $i^W_{p,r}$ be the formal identity mapping from W_p onto W_r (when $r = 2$ we abuse notation by also regarding the operator into $L_2(\nu)$). Sudakov's lemma [Su], stated as Proposition 4.1 in [BLM], gives the entropy estimate

$$\log E(B(W_p), \|\cdot\|_{L_2(\nu)}, t) \le C\left(\frac{\ell(i^{W\,*}_{p,2})}{t}\right)^2 .$$

The Pajor-Tomczak lemma [PT-J], stated as Proposition 4.2 in [BLM], gives the entropy estimate

$$\log E(B(W_2), \|\cdot\|_{L_p(\nu)}, t) \le C\left(\frac{\ell(i^W_{2,p})}{t}\right)^2 .$$

The ideal properties of ℓ imply that if ν (respectively, μ) is a probability measure on the finite set A (respectively, B), and Y (respectively W) is a space of scalar functions on A (respectively B), and $v\colon Y \to W$ is a linear operator which is an isometry from Y_p onto W_p and has norm at most C as an operator from Y_2 into W_2, then $\ell(i^{W\,*}_{p,2}) \le C\ell(i^{Y\,*}_{p,2})$.

The entropy lemma we use is a variation on Propositions 4.6 and 7.2 in [BLM]. The result we need later is different from that in [BLM] since we cannot replace the subspace X of $L_p(\mu)$ by an (isomorphic or even isometric) copy of X in $L_p(\nu)$ but rather must move all of $L_p(\mu)$ isometrically onto $L_p(\nu)$. Moreover, formally speaking, Proposition 7.2 is only partly proved in [BLM] and contains some unclear statements (e.g., the claim in the sentence immediately following (7.11) seems formally wrong and should be adjusted slightly). The accumulation of the adjustments needed to obtain Proposition 2.1 below from the arguments in [BLM] required some effort on our part, so we judged it worthwhile to outline proofs of the entropy estimates we need.

PROPOSITION 2.1: *Let X be an n-dimensional subspace of $L_p(\overline{N}, \mu)$ for some probability measure μ on $\overline{N} = \{1, \dots, N\}$. Then there is a density α on $(\overline{N}, \mu)$ satisfying the following: Put $\tilde{X} = \{x/\alpha^{\frac{1}{p}}\colon x \in X\}$ and let $B(\tilde{X}_r)$ be the closed unit ball of $\tilde{X}$ in $L_r(\overline{N}, \alpha\, d\mu)$. Then for some constant C,*

(i) $\log E(B(\tilde{X}_p), \|\cdot\|_\infty, t) \le C(p-1)^{\frac{p-2}{2}} n(\log n)^{1-\frac{p}{2}}(\log N)^{\frac{p}{2}} t^{-p}$, *for* $1 < p < 2$.

(ii) $\|f\|_{L_\infty} \le (2n)^{1/p} \|f\|_{L_p(\alpha d\mu)}$, for $1 < p < 2$.

(iii) $\log E(B(\tilde{X}_p), \|\cdot\|_\infty, t) \le C(\log N) n t^{-2}$, *for* $2 \le p < \infty$.

(iv) $\|f\|_{L_\infty} \le (2n)^{1/2} \|f\|_{L_p(\alpha d\mu)}$, *for* $2 < p < \infty$.

Proof: It is easy to reduce to the case of measures which are strictly positive (i.e., for which all points of $\overline{N}$ have positive μ measure). The conclusion is invariant under change of density of the original measure, so we can assume, without loss of generality, that μ is the uniform measure on $\overline{N}$ (this simplifies slightly the notation below).

Lewis [L] showed that there is a density β on $\overline{N}$ and an orthonormal (in $L_2(\beta d\mu)$) basis $f_1, \dots, f_n$ for $Y = \{x/\alpha_1^{1/p} : x \in X\}$ so that $\sum_{i=1}^n f_i^2 = n$.

The density α is $\frac{\beta+1}{2}$. Then $\tilde{X}$ consists of all vectors of the form $v(y)$ with y in Y, where $v(y) = \left(\frac{\beta}{\alpha}\right)^{1/p} y$. The linear operator v defines an isometry from $L_p(\beta d\mu)$ onto $L_p(\alpha d\mu)$ and has norm at most $2^{1/p}$ as an operator from $L_2(\beta d\mu)$ into $L_2(\alpha d\mu)$. As mentioned before the statement of Proposition 2.1, Sudakov's lemma gives the entropy estimate

$$\log E(B(\tilde{X}_p), \|\cdot\|_{L_2(\alpha\mu)}, t) \le C t^{-2} p n K(X)^2.$$

Using the fact that $(\frac{\beta}{\alpha})^{1/2} f_1, \dots, (\frac{\beta}{\alpha})^{1/2} f_n$ is orthonormal in $L_2(\alpha d\mu)$ and the Maurey-Khintchine inequality, we get for all $1 \le q < \infty$:

$$\begin{aligned} \ell^2(i_{2,q}^{\tilde{X}}) &= \mathbb{E}\left\| \sum_{i=1}^n g_i \left(\frac{\beta}{\alpha}\right)^{1/2} f_i \right\|_{L_q(\alpha d\mu)}^2 \\ &\le Cq \left\| (\sum_{i=1}^n \left(\frac{\beta}{\alpha}\right) f_i^2)^{1/2} \right\|_{L_q(\alpha d\mu)}^2 \\ &\le 2Cqn. \end{aligned}$$

As mentioned before the statement of Proposition 2.1, the Pajor-Tomczak lemma gives the entropy estimate

$$(+) \qquad \log E(B(\tilde{X}_2), \|\cdot\|_{L_q(\alpha d\mu)}, t) \le C t^{-2} q n.$$

Pick $q = \log 2N$; then, since $\alpha > 1/2$, $\|\cdot\|_\infty \le e\|\cdot\|_{L_q(\alpha d\mu)}$. Since $B(\tilde{X}_p) \subset B(\tilde{X}_2)$ for $p \ge 2$, this gives (iii). The Lewis change of density forces, for f in Y,

$\|f\|_{L_\infty} \le n^{1/2}\|f\|_{L_p(\beta d\mu)}$ (see e.g. Lemma 7.1 in [BLM]). Since $\frac{\beta}{\alpha} \le 2$, we have for f in $\tilde{X}$ that $\|f\|_{L_\infty} \le (2n)^{1/2}\|f\|_{L_p(\alpha d\mu)}$. This gives (iv).

To deal with the case $1 < p < 2$, we refer to the proof of Proposition 7.2 (ii) in [BLM]. By applying Hölder's inequality and a clever duality argument, one obtains formally from (+), for $1 \le t \le 2n$, that

$$(++) \qquad \log E(B(\tilde{X}_p), \|\cdot\|_{L_2(\alpha d\mu)}, t) \le C(p-1)^{-1}\left(\frac{C}{t}\right)^{2p/(2-p)} n \log n.$$

Using, for $1 < s < t$, the obvious inequality

$$\log E(B(\tilde{X}_p), \|\cdot\|_{L_\infty}, t) \le \log E(B(\tilde{X}_p), \|\cdot\|_{L_2(\alpha d\mu)}, s) + \log E(B(\tilde{X}_2), \|\cdot\|_{L_\infty}, t/s),$$

(++), and (iii) in the statement of the proposition, we obtain (i) for $1 \le t \le 2n$ by minimizing over s. Now the Lewis change of density forces, for f in Y, $\|f\|_{L_\infty} \le n^{1/p}\|f\|_{L_p(\beta d\mu)}$. Since $\frac{\beta}{\alpha} \le 2$, we have for f in $\tilde{X}$ that $\|f\|_{L_\infty} \le (2n)^{1/p}\|f\|_{L_p(\alpha d\mu)}$. This gives (ii) as well as (i) when $t > 2n$. ■

The following proposition and its proof is an adjustment of results from Talagrand's paper [T]. (The idea of "splitting the large atoms", used also in [T], is due to the authors.)

PROPOSITION 2.2: *Let X be an n-dimensional subspace of $L_p(\overline{N}, \tau)$ for some probability measure τ on $\overline{N} = \{1, \dots, N\}$. Then there are $N \le M \le \frac{3}{2}N$ and a probability measure ν on $\overline{M} = \{1, \dots, M\}$ so that:*

(i) *There is a partition $\{\sigma_1, \dots, \sigma_n\}$ of $\overline{M}$ with $\sum_{i\in\sigma_j} \nu\{i\} = \tau\{j\}$ for $j = 1, \dots, n$.*

(ii) $\mathbb{E}\sup\left\{\left|\sum_{i=1}^M g_i \nu\{i\}|y_i|^p\right| : y \in Y, \|y\| \le 1\right\} \le$

$$C(p-1)^{\frac{p-2}{4}}\left(\frac{n}{N}\right)^{\frac{1}{2}}(\log n)^{\frac{6-p}{4}}(\log N)^{\frac{p}{4}},$$

for $1 < p < 2$, where $y_1, \dots, y_n$ are the coordinates of the vector y and Y is the image of X under the natural isometry J_p from $L_p(\overline{N}, \mu)$ into $L_p(\overline{M}, \nu)$, defined by $(J_p x)_i = x_j$ if $i \in \sigma_j$.

(iii) $\mathbb{E}\sup\left\{\left|\sum_{i=1}^M g_i \nu\{i\}|y_i|^p\right| : y \in Y, \|y\| \le 1\right\} \le C_p n^{p/4} N^{-1/2} \log n (\log N)^{1/2}$, *for $2 < p < \infty$. C_p can be taken to be $Cp^2 2^{p/2}$.*

Proof: It is easy to reduce to the case of measures which are strictly positive. Next, note that if the proposition is true for *one* strictly positive probability measure on $\overline{N}$, then it is true for all of them. This is because the left hand side of (ii) is invariant under a change of density ϕ if we replace the subspace Y of $L_p(\nu)$ with its image under the natural isometry from $L_p(\nu)$ onto $L_p(\phi\, d\nu)$, defined by $Tf = f/\phi^{1/p}$. Thus we can assume that τ is the measure $\alpha d\mu$ given by the conclusion of Proposition 2.1.

Splitting the atoms of τ of mass larger than $4/N$ into pieces each of size between $2/N$ and $4/N$ produces $\overline{M}$, the measure ν, and, *a fortiori*, the space Y along with the isometry $J = J_p$; (i) is thus satisfied. Since J also defines an isometry J_r from $L_r(\overline{N}, \tau)$ into $L_r(\overline{M}, \nu)$ for all $0 < r \le \infty$, the conclusion of Proposition 2.1 remains true for the measure space $(\overline{M}, \nu)$ (where of course $\tilde{X}$ is replaced by Y).

Let δ be the natural distance associated with the Gaussian process appearing in (ii), defined for y, z in Y by

$$\delta(y,z) = \left(\sum_{i=1}^{M} [\nu\{i\}(|y_i|^p - |z_i|^p)]^2\right)^{1/2}.$$

Let $1 < p < 2$, fix y, z in $B(Y_p)$, and set $u_i = |y_i| \vee |z_i|$. Then

$$\begin{aligned}
\delta(y,z)^2 &\le \sum_{i=1}^{M} \nu\{i\}^2 p^2 u_i^{2p-2} |y_i - z_i|^2 \\
&\le \|y-z\|_\infty^p \, 4p^2 N^{-1} \sum_{i=1}^{M} \nu\{i\} u_i^{2p-2} |y_i - z_i|^{2-p} \\
&\le 4p^2 N^{-1} \|y-z\|_\infty^p \left(\sum_{i=1}^{M} \nu\{i\} u_i^p\right)^{2(p-1)/p} \left(\sum_{i=1}^{M} \nu\{i\} |y_i - z_i|^p\right)^{(2-p)/p} \\
&\le 2^6 N^{-1} \|y-z\|_\infty^p .
\end{aligned}$$

Thus by Proposition 2.1 (ii) we get that the δ-diameter of $B(Y_p)$ is less than $2^4 n^{1/2} N^{-1/2}$ and from Proposition 2.1 (i) that:

$$\begin{aligned}
\log E(B(Y_p), \delta, t) &\le \log E(B(Y_p), \|\cdot\|_\infty^{p/2}, 2^{-3} N^{1/2} t) \\
&\le \log E(B(Y_p), \|\cdot\|_\infty, 2^{-6/p} N^{1/p} t^{2/p}) \\
&\le C(p-1)^{\frac{p-2}{2}} n (\log n)^{1-\frac{p}{2}} (\log N)^{\frac{p}{2}} N^{-1} t^{-2}.
\end{aligned}$$

The last inequality in this last display requires $t \ge 2^3 N^{-1/2}$; for $0 < t < 2^3 N^{-1/2}$ use volume considerations in the n-dimensional space $B(Y_\infty)$ to get

$$\begin{aligned}
\log E(B(Y_p), \delta, t) &\le \log E(B(Y_p), \|\cdot\|_\infty, 1) + \log E(B(Y_\infty), \|\cdot\|_\infty, 2^{-6/p} N^{1/p} t^{2/p}) \\
&\le C(p-1)^{\frac{p-2}{2}} n (\log n)^{1-\frac{p}{2}} (\log N)^{\frac{p}{2}} + Cn \log(CN^{-1} t^{-2}).
\end{aligned}$$

By Dudley's theorem (see, e.g., [MP, p. 25]),

$$\begin{aligned}
&\mathbb{E}\sup\Big\{\big|\sum_{i=1}^{M} g_i\nu\{i\}|y_i|^p\big| : y\in Y,\quad \|y\|\le 1\Big\}\\
&\quad\le 2^4 n^{1/2}N^{-1/2} + C(p-1)^{\frac{p-2}{4}} n^{1/2}(\log n)^{\frac{2-p}{4}}(\log N)^{\frac{p}{4}}N^{-1/2}\\
&\quad + Cn^{1/2}\int_0^{2^3N^{-1/2}} \log^{1/2}(CN^{-1}t^{-2})\,dt\\
&\quad + C(p-1)^{\frac{p-2}{4}} n^{1/2}(\log n)^{\frac{2-p}{4}}(\log N)^{\frac{p}{4}}N^{-1/2}\int_{2^3N^{-1/2}}^{2^4n^{1/2}N^{-1/2}} t^{-1}\,dt\\
&\quad\le C(p-1)^{\frac{p-2}{4}} n^{1/2}(\log n)^{\frac{6-p}{4}}(\log N)^{\frac{p}{4}}N^{-1/2}.
\end{aligned}$$

This proves (ii).

To prove (iii), assume now $2<p<\infty$. Fix y,z in $B(Y_p)$, and set $u_i=|y_i|\vee|z_i|$. Then

$$\begin{aligned}
\delta(y,z)^2 &\le \sum_{i=1}^{M}\nu\{i\}^2p^2u_i^{2p-2}|y_i-z_i|^2\\
&\le \|y-z\|_\infty^2\,4p^2N^{-1}\sum_{i=1}^{M}\nu\{i\}u_i^{2p-2}\\
&\le 4p^2N^{-1}\,\|y-z\|_\infty^2\,\|u\|_\infty^{p-2}\sum_{i=1}^{M}\nu\{i\}u_i^p\\
&\le 4p^2 2^{p/2}\frac{n^{(p-2)/2}}{N}\,\|y-z\|_\infty^2,
\end{aligned}$$

where the last inequality follows from Proposition 2.1 (iv). Thus the δ diameter of $B(Y_p)$ is less than $4p2^{p/4}n^{p/4}N^{-1/2}$ and Proposition 2.1 (iii) implies:

$$\begin{aligned}
\log E(B(Y_p),\delta,t) &\le \log E(B(Y_p),\|\cdot\|_\infty, p^{-1}2^{-(p+4)/4}n^{-(p-2)/4}N^{1/2}t)\\
&\le Cp^2 2^{p/2}n^{p/2}N^{-1}(\log N)t^{-2}
\end{aligned}$$

as long as $t\ge p2^{(p+4)/4}n^{(p-2)/4}N^{-1/2}$. For smaller t we get by the usual volume considerations,

$$\begin{aligned}
&\log E(B(Y_p),\delta,t)\\
&\quad\le \log E(B(Y_p),\|\cdot\|_\infty,1) + \log E(B(Y_\infty),\|\cdot\|_\infty, p^{-1}2^{-(p+4)/4}n^{-(p-2)/4}N^{1/2}t)\\
&\quad\le Cp^2 2^{p/2}n^{p/2}N^{-1}(\log N) + Cn\log(Cp2^{p/4}n^{(p-2)/4}N^{-1/2}t^{-1}).
\end{aligned}$$

By Dudley's theorem,

$$\begin{aligned}
&\mathbb{E}\sup\Big\{\big|\sum_{i=1}^{M} g_i\nu\{i\}|y_i|^p\big| : y\in Y,\quad \|y\|\le 1\Big\}\\
\le& Cp2^{p/4}n^{p/4}N^{-1/2} + Cp^2 2^{p/2}n^{(p-1)/2}N^{-1}(\log N)^{1/2}\\
&+ Cn^{1/2}\int_0^{p2^{(p+4)/4}n^{(p-2)/4}N^{-1/2}} \log^{1/2}(Cp2^{p/4}n^{(p-2)/4}N^{-1/2}t^{-1})\,dt\\
&+ Cp2^{p/4}n^{p/4}N^{-1/2}(\log N)^{1/2}\int_{p2^{(p+4)/4}n^{(p-2)/4}N^{-1/2}}^{4p2^{p/4}n^{p/4}N^{-1/2}} t^{-1}\,dt.
\end{aligned}$$

For a fixed p the last term is dominating and one gets

$$\mathbb{E}\sup\Big\{\big|\sum_{i=1}^{M} g_i\nu\{i\}|y_i|^p\big| : y\in Y,\quad \|y\|\le 1\Big\} \le C_p n^{p/4}N^{-1/2}\log n(\log N)^{1/2}$$

where C_p can be taken to be $Cp^2 2^{p/2}$. ■

COROLLARY 2.3: *Let X be an n-dimensional subspace of $L_p(\overline{N},\tau)$ for some probability measure τ on $\overline{N}=\{1,\dots,N\}$ and let $L_p(\overline{M},\nu)$, J, and Y be given from Proposition 2.2. Then there is a partition $M_1\cup M_2$ of $\overline{M}$ into two sets of cardinality at most $\frac{7}{8}N$ such that for each y in Y and $j=1,2$:*

(i) $\|1_{M_j}y\|^p_{L_p(\overline{M},\nu)} \le$

$$\Big(1/2 + C(p-1)^{\frac{p-2}{4}}\big(\frac{n}{N}\big)^{\frac12}(\log n)^{\frac{6-p}{4}}(\log N)^{\frac{p}{4}}\Big)\|y\|^p_{L_p(\overline{M},\nu)},$$

when $1<p<2$; while

(ii) $$\|1_{M_j}y\|^p_{L_p(\overline{M},\nu)} \le \left(1/2 + C_p\left(\frac{n^{\frac{p}{2}}}{N}\right)^{\frac12}\log n(\log N)^{\frac12}\right)\|y\|^p_{L_p(\overline{M},\nu)},$$

for $2<p<\infty$.

Moreover, (i) and (ii) hold for most such partitions of $\overline{M}$.

Proof: First, notice that (ii) in Proposition 2.2 still holds if we substitute independent Rademacher functions for the Gaussian variables g_i (and replace C by, e.g., $\sqrt{\frac{\pi}{2}}C$). This follows from a standard contraction principle. Consequently, if we again enlarge C,

$$\sup\Big\{\big|\sum_{i=1}^{M}\epsilon_i\nu\{i\}|y_i|^p\big| : y\in Y, \|y\|\le 1\Big\} \le C(p-1)^{\frac{p-2}{4}}\left(\frac{n}{N}\right)^{\frac12}(\log n)^{\frac{6-p}{4}}(\log N)^{\frac{p}{4}}$$

holds for most choices of signs $\epsilon_i = \pm 1$. Since also for most choices of signs the difference between the number of plus signs and minus signs is less than $M/8$, (i) follows. (ii) follows similarly. ■

III. Computing p-summing norms

Given a linear operator $u\colon X \to Y$ of finite rank, $1 \le q \le \infty$, and positive integers n, k, define

$$\nu_q^{(n,k)}(u) = \inf\left\{ \sum_{i=1}^{k} \nu_q^{(n)}(u_i) \colon u = \sum_{i=1}^{k} u_i \right\},$$

where

$$\nu_q^{(n)}(v) = \inf\left\{ \|A\|\,\|w\|\,\|B\| \,;\ A\colon X \to \ell_\infty^n; w\colon \ell_\infty^n \to \ell_q^n \text{ diagonal}, B\colon \ell_q^n \to Y, v = BwA \right\}.$$

In Tomczak's terminology [T-J, p. 181], $\nu_q^{(n,1)} = \nu_q^{(n)}$, while $\lim_{k\to\infty} \nu_q^{(n,k)} = \hat{\nu}_q^{(n)}$ gives the cogradation which is dual to the natural gradation $\pi_p^{(n)}$ of the p-summing norm [T-J, Theorem 24.2] (or something like that!).

PROPOSITION 3.1: *Let $n \le N$ be positive integers; $u\colon X \to Y$ a linear operator with X finite dimensional and* $\dim(Y) \le n$. *Then, putting* $q = p/(p-1)$,

(i) *For* $1 < p < 2$,

$$\nu_q^{(\frac{7}{8}N,2)}(u) \le \left(1 + C(p-1)^{\frac{p-2}{4}} \left(\frac{n}{N}\right)^{\frac{1}{2}} (\log n)^{\frac{6-p}{4}} (\log N)^{\frac{p}{4}}\right) \nu_q^{(N,1)}(u).$$

(ii) *For* $2 < p < \infty$,

$$\nu_q^{(\frac{7}{8}N,2)}(u) \le \left(1 + C_p \left(\frac{n^{\frac{p}{2}}}{N}\right)^{\frac{1}{2}} \log n (\log N)^{\frac{1}{2}}\right) \nu_q^{(N,1)}(u).$$

Proof: For some probability measure τ on $\overline{N}$, we can take $A\colon Y^* \to L_p(\overline{N},\tau)$, $B\colon L_1(\overline{N},\tau) \to X^*$, so that $\|A\|\,\|B\| = \nu_q^{(N)}(u)$ and $u^* = Bi_{p,1}A$. Apply Proposition 2.2 to the subspace AY of $L_p(\overline{N},\tau)$ to get the measure space $L_p(\overline{M},\nu)$ and the natural isometric embedding $J_p\colon L_p(\overline{N},\tau) \to L_p(\overline{M},\nu)$. By Corollary 2.3, we get a partition $M_1 \cup M_2$ of $\overline{M}$ into two sets of cardinality at most $\frac{7}{8}N$ such that for each y in Y, $j = 1,2$, and in the case $1 < p < 2$:

$$\left\|1_{M_j} J_p A y\right\|_{L_p(\overline{M},\nu)}^p \le \left(1/2 + C(p-1)^{\frac{p-2}{4}} \left(\frac{n}{N}\right)^{\frac{1}{2}} (\log n)^{\frac{6-p}{4}} (\log N)^{\frac{p}{4}}\right) \|Ay\|_{L_p(\overline{N},\tau)}^p .$$

Denote for $j = 1,2$ the injection from $L_p(M_j, \nu_{|M_j})$ to $L_1(M_j, \nu_{|M_j})$ by $i^j_{p,1}$ and let P be the conditional expectation projection from $L_1(\overline{M}, \nu)$ onto $J_1[L_1(\overline{N}, \tau)]$ followed by J_1^{-1}. Thus $u^* = BPi^1_{p,1}1_{M_1}J_1A + BPi^2_{p,1}1_{M_2}J_1A$ and

$$\begin{aligned}
\nu_q^{(\frac{7}{8}N,2)}(u) &\le \sum_{j=1}^{2} \nu_q^{(\frac{7}{8}N)}([BPi^j_{p,1}1_{M_j}J_1A]^*) \\
&\le \sum_{j=1}^{2} \|1_{M_j}J_1A\|\,\|i^j_{p,1}\|\|BP\| \\
&\le \left(\frac{1}{2} + C(p-1)^{\frac{p-2}{4}}(\frac{n}{N})^{\frac{1}{2}}(\log n)^{\frac{6-p}{4}}(\log N)^{\frac{p}{4}}\right)^{\frac{1}{p}} \|A\|\,\|B\| \sum_{j=1}^{2} \nu(M_j)^{\frac{1}{q}} \\
&\le \|A\|\,\|B\| \left(1 + 2C(p-1)^{\frac{p-2}{4}}(\frac{n}{N})^{\frac{1}{2}}(\log n)^{\frac{6-p}{4}}(\log N)^{\frac{p}{4}}\right)^{\frac{1}{p}} \\
&\le \|A\|\,\|B\| \left(1 + 2C(p-1)^{\frac{p-2}{4}}(\frac{n}{N})^{\frac{1}{2}}(\log n)^{\frac{6-p}{4}}(\log N)^{\frac{p}{4}}\right).
\end{aligned}$$

This completes the proof when $1 < p < 2$; the other case is similar. ■

THEOREM 3.2: *Suppose that* $\dim(X) \le n$, u: $X \to Y$ *is a linear operator and* $\epsilon > 0$. *Then,*

$$\pi_p(u) \le (1+\epsilon)\pi_p^{(m)}(u),$$

as long as

(i) $1 < p < 2$ *and* $m \ge K(p-1)^{\frac{p-2}{2}}\epsilon^{-2}n(\log n)^{\frac{6-p}{2}}\left(\log\left((p-1)^{\frac{p-2}{2}}\epsilon^{-2}n\right)\right)^{\frac{p}{2}}$ *for some absolute constant* K, *or*

(ii) $2 < p < \infty$ *and* $m \ge K_p\epsilon^{-2}n^{\frac{p}{2}}(\log n)^2\log(\epsilon^{-2}n^{\frac{p}{2}})$.

Proof: Without loss of generality, we can assume that $\dim(Y) \le n$. By duality [T-J, Theorem 24.2], it is enough to prove that

$$\hat{\nu}_q^{(m)}(v) \le (1+\epsilon)\nu_q^N(v)$$

for all v: $Y \to X$ and all positive integers $N \ge n$. Iterating Proposition 3.1, we get for all k (with $(\frac{7}{8})^k N \ge n$) and for $1 < p < 2$ that

$$\nu_q^{([\frac{7}{8}]^k N,2^k)}(u) \le \prod_{j=1}^{k}\left(1 + C(p-1)^{\frac{p-2}{4}}\left(\frac{n}{(\frac{7}{8})^{j-1}N}\right)^{\frac{1}{2}}(\log n)^{\frac{6-p}{4}}(\log\left[(\tfrac{7}{8})^{j-1}N\right])^{\frac{p}{4}}\right)\nu_q^{(N)}(u).$$

The product on the right hand side of the above inequality is smaller than $1+\epsilon$ as long as

$$(p-1)^{\frac{p-2}{4}}\left(\frac{n}{(\frac{7}{8})^k N}\right)^{\frac{1}{2}}(\log n)^{\frac{6-p}{4}}(\log\left[(\tfrac{7}{8})^k N\right])^{\frac{p}{4}} \leq \delta\epsilon,$$

(where $\delta=\delta(C)$ is an appropriate positive constant). Put $m=(\frac{7}{8})^k N$; then, as long as $m \geq \delta'(p-1)^{\frac{p-2}{2}}\epsilon^{-2}n(\log n)^{\frac{6-p}{2}}\left(\log\left((p-1)^{\frac{p-2}{2}}\epsilon^{-2}n\right)\right)^{p/2}$,

$$\hat{\nu}_q^{(m)}(u) \leq \nu_q^{(m,2^k)}(u) \leq \nu_q^{(N)}(u).$$

This completes the proof when $1<p<2$; the case $2<p<\infty$ is similar. ■

Remark: As we have presented it, the proof of Theorem 3.2 does not recapture the result of Szarek mentioned in the introduction. Actually, our approach does work when $p=1$ and the technical difficulties are easier in this case because the entropy considerations of Section II are not needed.

IV. Examples and concluding remarks

For $p>2$, $\pi_p^{(k)}(\ell_2^n) \leq k^{\frac{1}{p}}$, while $\pi_p(\ell_2^n) \geq \sqrt{n/p}$ [T-J, Theorem 10.2]. Consequently, Theorem 3.2 is precise except for the $\log n$ terms.

It is natural to ask what value of k is needed for $\pi_p^{(k)}(u)$ to well-estimate $\pi_p(u)$ for a general operator u of rank n. When $p=1$, Figiel–Pełczynski [T-J, p.184] checked that k must be exponential in n. The authors and J. Bourgain checked that a result of Bourgain's [B] yields that for $1<p\neq 2<\infty$, k grows faster than any power of n.

PROPOSITION 4.1: *Let $1<p\neq 2<\infty$ and $C<\infty$. Suppose that for each $s=1,2,\ldots$, k_s satisfies*

$$\pi_p(u) < C\pi_p^{(k_s)}(u)$$

for all operators u of rank at most s. Then for all $K<\infty$, $k_s s^{-K}\to\infty$ as $N\to\infty$.

Proof: Fix $1<p\neq 2<\infty$, K, C, and let $\delta>0$ with $\delta K<1$. Given $N=2^n$ for some n, we identify L_p^N with $L_p(G)$, where G is the group $\{-1,1\}^n$ with normalized Haar measure, dg.

Let $E = \operatorname{span}\{w_S : |S| \geq n - m\}$ where $\frac{m}{n} \log \frac{n}{m} \sim \delta$; so $\dim E \sim (\frac{n}{m})^m < N^\delta$. Here we follow Bourgain's notation [B]; for $S \subset \{1, \ldots, n\}$, $w_S = \prod_{i \in S} r_i$, with r_i the i-th coordinate projection (Rademacher) on G. Let $j^E_{\infty,p}$ be the formal identity from E_∞ to L_p^N. We shall use Bourgain's result [B] that if T is an operator on L_p^N which is the identity on E and $\|T\| < C$, then $\operatorname{trace} T \sim N$ (meaning $|\operatorname{trace}(I - T)| = o(N)$), to prove that if

$$\hat{N}_p^{(k)}(j^E_{\infty,p}) < C\nu_p(j^E_{\infty,p}) \quad (= C),$$

then for large N, $k > N^{\delta K} \geq (\dim E)^K$. This gives the dual form of the conclusion of Proposition 4.1.

For notational convenience, set $\alpha = j^E_{\infty,p}$ and suppose that for certain k we have $\alpha = \sum_i \alpha_i$ with $\sum_i \nu_p^k(\alpha_i) < C$. This means that there are factorizations

$$E \xrightarrow{\tau_i} \ell_\infty^k \xrightarrow{\Delta_i} \ell_p^k \xrightarrow{\gamma_i} L_p^N$$

of α_i with $\|\tau_i\| = \|\Delta_i\| = 1$, Δ_i diagonal, and $\sum_i \left\|\gamma_i\right\| < C$. This diagram also gives that $\sum_i \nu_1(\alpha_i) < Ck^{1-\frac{1}{p}}$. Extend τ_i to a map $\tilde{\tau}_i : L_\infty^N \to \ell_\infty^k$ with $\|\tilde{\tau}_i\| = 1$, set $\tilde{\alpha}_i = \gamma_i \Delta_i \tilde{\tau}_i$, and let $\tilde{\alpha} = \sum_i \tilde{\alpha}_i$. Then

$$\nu_1(\tilde{\alpha}) \leq \sum_i \nu_1(\tilde{\alpha}_i) < Ck^{1-1/p} \quad \text{and} \quad \pi_p(\tilde{\alpha}) = \nu_p(\tilde{\alpha}) < C.$$

Now replace $\tilde{\alpha}$ by its average β over the group G, defined by

$$\beta = \int_G T_g \tilde{\alpha} T_g \, \mathrm{d}g \quad (g^{-1} = g \text{ in } G).$$

The operator β is translation invariant (a multiplier) and satisfies the same conditions as $\tilde{\alpha}$; namely,

$$\beta_{|E} = \alpha, \quad \nu_1(\beta) < Ck^{1-\frac{1}{p}}, \quad \pi_p(\beta) < C.$$

Since β is translation invariant, Haar measure on G is a suitable Pietsch measure, which means that $\|\beta i_{p,\infty}\| < C$. Thus $\operatorname{trace}(\beta i_{p,\infty}) \sim N$ by Bourgain's result [B]. However,

$$|\operatorname{trace}(\beta i_{p,\infty})| \leq \nu_1(\beta i_{p,\infty}) \, \|i_{p,\infty}\| < Ck^{1-1/p} N^{1/p},$$

which is $o(N)$ if $k \leq N^{\delta K}$. ■

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